Dependence on parameters for discrete second order boundary value problems

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Abstract

We investigate the dependence on parameters for second order difference equations with two point boundary value conditions by using a variational method in case when the corresponding Euler action functional is coercive. Some applications for discrete Emden-Fowler equation are also given.

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1 Introduction

The variational approach towards the existence of solutions to nonlinear difference equations received some serious attention, see for example, [4], [1], [2], [8], [11], [13]. Various types of methods have so far been employed, in

fact the approaches valid for boundary value problems for differential equations have successfully been adapted and somehow extended due to the fact that in the setting of difference equations the boundary value problems are considered in a finite dimensional space.

In this work we mainly intend to investigate the dependence on a functional parameter u for the coercive second order boundary value problems taking as an example the problem originally considered in [8] by variational method and in [10] by the lower-upper function method. Later such boundary value problem has been reconsidered in a variational formulation in [3] with weaker assumptions than those of [8]. However, both the approach of [8] and the topological method from [10] yield the same existence result - with the same assumptions as is shown in [3] - it is the variational method which, in our opinion, allows for considering the dependence of the solution on a parameter in some systematic way.

The approach towards investigation of a dependence on a functional parameter for solutions of ODE in case of coercive action functional originates for example from [7]. We also base on some of ideas from [7] but we put them in a different context and for a discrete problem. Such an investigation has not been undertaken yet to the best of our knowledge. The idea of the continuous dependence on parameters could be summarized as follows: we consider a discrete boundary value problem which is subject to certain (functional) parameter and which has a solution with respect to any parameter (function). Therefore corresponding to a sequence of parameters there exists a sequence of solutions. Supposing that the sequence of parameters is convergent (in a suitable sense) we arrive at the limit of a sequence of solutions, which itself is a solution to the considered problem corresponding to the limit of the parameter sequence. What is important and what constitutes the main point is that all solutions, both in the sequence and the limit one, share the same properties.

2 Dependence on parameters for second order coercive problem

Before we provide the statement of the problem under consideration, we introduce some notation. In what follows by $[A, B]$ we mean the discrete interval $\{A, ..., B\}$. $C([A, B], R)$ is a space of functions $u : [A, B] \to R$

(defined on a discrete interval, and thus necessarily continuous) equipped with classical maximum norm $\|u\|_C = \max_{k\in\{A,...,B\}} |u(k)|$.

Let $M > 0$ be fixed. A parameter function u belongs to

$$L_M = \{u \in C([1,T], R) : \|u\|_C \leq M\}.$$

Δ denotes the forward difference operator, i.e. $\Delta x(k) = x(k+1) - x(k)$. E stands for the space of functions $y : [0, T+1] \to R$ such that $y(0) = y(T+1) = 0$ considered with norm $\|y\| = \sqrt{\sum_{k=1}^{T} (\Delta y(k))^2}$. By $|\cdot|$ we denote Euclidean norm on E and we see that

$$\gamma |y| \leq \|y\| \leq \gamma_1 |y| \text{ for all } y \in E \tag{1}$$

for certain constants $\gamma, \gamma_1 > 0$ which do not depend on y.

In this section we will investigate the following problem in E

$$\Delta(p(k)\Delta x(k-1)) + f(k, x(k), u(k)) = g(k) \tag{2}$$

which is subject to a parameter $u \in L_M$ and which satisfies the Dirichlet boundary conditions

$$x(0) = x(T+1) = 0. \tag{3}$$

We will assume that

A1 $f \in C([1,T] \times R \times [-M, M], R)$, $p \in C([1, T+1], R)$, $g \in C([1,T], R)$;

A2 there exists $\alpha > 0$ such that $yf(k, y, u) \leq 0$ for all $|y| \geq \alpha$, $|u| \leq M$ and $k = 1, ..., T$;

A3 $m = \min_{k\in\{1,...,T+1\}} p(k) > 0$.

Here $f \in C([1,T] \times R \times [-M, M], R)$ means that for each $k \in \{1, 2, ..., T\}$ function $f(k, \cdot, \cdot)$ is continuous on $R \times [-M, M]$. Let for $y \in E$

$$F(k, y(k), u(k)) = \int_0^{y(k)} f(k, t, u(k))\, dt.$$

With assumptions **A1**-**A3** the action functional $J : E \to R$ corresponding to (2)-(3) with a fixed function $u \in L_M$ reads

$$J_u(y) = \sum_{k=1}^{T+1} \left[\frac{p(k)}{2} \Delta y^2(k-1)\right] - \sum_{k=1}^{T} F(k, y(k), u(k)) + \sum_{k=1}^{T} g(k) y(k) \tag{4}$$

and it is coercive and continuous on E. Since it is obviously differentiable in the sense of Gâteaux with bounded Gâteaux variation at each point, it admits at least one minimizer satisfying (2)-(3), see [3], [8] for details. Namely, for any fixed $u \in L_M$ the set which consists of the arguments of a minimum to J_u

$$V_u = \left\{ x \in E : J_u(x) = \inf_{v \in E} J_u(v) \text{ and } \frac{d}{dx} J_u(x) = 0 \right\}$$

is non-empty. We will investigate the behavior of the sequence $\{x_n\}_{n=1}^{\infty}$ of solutions to (2)-(3) depending on the convergence of the sequence of parameters $\{u_n\}_{n=1}^{\infty}$. Moreover, we consider the case of the existence and dependence on parameters for positive solutions. Next, we investigate some general stability results in a sense which we describe later. In fact the dependence on a parameter is obtained as a special case of stability which we show by giving the alternative proof of the main result, namely Theorem 1.

We would like to mention that typically with (2)-(3) it is associated the following functional instead of (4)

$$J_u^1(y) = \sum_{k=1}^{T+1} \left[\frac{p(k)}{2} \Delta y^2 (k-1) - F(k, y(k), u(k)) + g(k) y(k) \right].$$

However, it requires that g, $f \in C([1, T+1], R)$. As in [8] we can show that (2)-(3) stands for critical point to (4) as well.

2.1 Dependence on parameters

Theorem 1 *Assume* ***A1-A3****. For any fixed $u \in L_M$ there exists at least one solution $x \in V_u$ to problem (2)-(3). Let $\{u_n\}_{n=1}^{\infty} \subset L_M$ be a convergent sequence of parameters, where $\lim_{n\to\infty} u_n = \overline{u} \in L_M$. For any sequence $\{x_n\}_{n=1}^{\infty}$ of solutions $x_n \in V_{u_n}$ to the problem (2)-(3) corresponding to u_n, there exist a subsequence $\{x_{n_i}\}_{i=1}^{\infty} \subset E$ and an element $\overline{x} \in E$ such that $\lim_{i\to\infty} x_{n_i} = \overline{x}$ and $J_{\overline{u}}(\overline{x}) = \inf_{y \in E} J_{\overline{u}}(y)$. Moreover, $\overline{x} \in V_{\overline{u}}$, i.e. $\overline{x}$ satisfies*

$$\Delta(p(k) \Delta \overline{x}(k-1)) + f(k, \overline{x}(k), \overline{u}(k)) = g(k), \ \overline{x}(0) = \overline{x}(T+1) = 0.$$

Proof. From [3] it follows that for each $n = 1, 2, ...$ there exists a solution $x_n \in V_{u_n}$ to (2)-(3) . We see that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Indeed, for any n we have $x_n \in V_{u_n} \subset \{x : J_{u_n}(x) \leq J_{u_n}(0)\}$. By **A2** we further

obtain for some $C>0$ and for all $x_n \in V_{u_n}$

$$\begin{aligned} &\sum_{k=1}^{T} F\left(k, x_n\left(k\right), u_n\left(k\right)\right) = \sum_{k=1}^{T} \int_0^{x_n(k)} f\left(k, t, u_n\left(k\right)\right) dt \leq \\ &\sum_{k=1}^{T} \int_{-\alpha}^{\alpha} \left|f\left(k, x_n\left(k\right), u_n\left(k\right)\right)\right| \leq C. \end{aligned} \tag{5}$$

Next, by (5) and by (1) we get

$$\begin{aligned} &J_{u_n}\left(x_n\right) = \sum_{k=1}^{T+1}\left[\tfrac{p(k)}{2}\Delta x_n^2\left(k-1\right)\right] - \sum_{k=1}^{T} F\left(k, x_n\left(k\right), u_n\left(k\right)\right) \\ &+\sum_{k=1}^{T} g\left(k\right) x_n\left(k\right) \geq \tfrac{m}{2}\left\|x_n\right\|^2 - \sqrt{\sum_{k=1}^{T} g^2\left(k\right)}\left|x_n\right| - C \geq \\ &\tfrac{m}{2}\left\|x_n\right\|^2 - \tfrac{1}{\gamma}\sqrt{\sum_{k=1}^{T} g^2\left(k\right)}\left\|x_n\right\| - C. \end{aligned} \tag{6}$$

On the other hand we see by definition of F that $-F\left(k, 0, u_n\left(k\right)\right) = 0$, so

$$J_{u_n}\left(x_n\right) \leq J_{u_n}\left(0\right) = 0.$$

Thus

$$\frac{m}{2}\left\|x_n\right\|^2 - \frac{1}{\gamma}\sqrt{\sum_{k=1}^{T} g^2\left(k\right)}\left\|x_n\right\| \leq C. \tag{7}$$

Since (7) treated as a quadratic inequality with variable $t = \|x_n\|$ has solutions in a bounded closed interval and since n was fixed arbitrarily, we see that $\{x_n\}_{n=1}^{\infty}$ is bounded in E. Hence, it has a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$. We denote its limit by $\overline{x}$. (We note that in [3] relation (6) is used in demonstrating that the action functional is indeed coercive.)

Now we demonstrate that $\overline{x}$ satisfies (2)-(3) corresponding to $\overline{u}$. Firstly, we observe that there exists $x_0 \in E$ such that x_0 solves (2)-(3) with $\overline{u}$ and $J_{\overline{u}}\left(x_0\right) = \inf_{y\in E} J_{\overline{u}}\left(y\right)$. We see that there are two possibilities: namely either $J_{\overline{u}}\left(x_0\right) < J_{\overline{u}}\left(\overline{x}\right)$ or $J_{\overline{u}}\left(x_0\right) = J_{\overline{u}}\left(\overline{x}\right)$. On the one hand we suppose that $J_{\overline{u}}\left(x_0\right) < J_{\overline{u}}\left(\overline{x}\right)$. Now there exists a constant $\delta > 0$ such that in fact

$$J_{\overline{u}}\left(\overline{x}\right) - J_{\overline{u}}\left(x_0\right) > \delta > 0. \tag{8}$$

We investigate the convergence of the right hand side of the inequality

$$\delta < \left(J_{u_{n_i}}\left(x_{n_i}\right) - J_{\overline{u}}\left(x_0\right)\right) - \left(J_{u_{n_i}}\left(x_{n_i}\right) - J_{u_{n_i}}\left(\overline{x}\right)\right) - \left(J_{u_{n_i}}\left(\overline{x}\right) - J_{\overline{u}}\left(\overline{x}\right)\right) \tag{9}$$

which is equivalent to (8). It is obvious, by continuity, that

$$\lim_{i\to\infty}\left(J_{u_{n_i}}\left(\overline{x}\right)-J_{\overline{u}}\left(\overline{x}\right)\right)=0 \text{ and } \lim_{i\to\infty}\left(J_{u_{n_i}}\left(x_{n_i}\right)-J_{u_{n_i}}\left(\overline{x}\right)\right)=0. \quad (10)$$

We also have

$$\lim_{i\to\infty}\left(J_{u_{n_i}}\left(x_0\right)-J_{\overline{u}}\left(x_0\right)\right)=0. \quad (11)$$

Since x_{n_i} minimizes $J_{u_{n_i}}$ over E we see that $J_{u_{n_i}}\left(x_{n_i}\right)\leq J_{u_{n_i}}\left(x_0\right)$ for any n_i. Therefore, we get by (11)

$$\lim_{i\to\infty}\left(J_{u_{n_i}}\left(x_{n_i}\right)-J_{\overline{u}}\left(x_0\right)\right)\leq\lim_{i\to\infty}\left(J_{u_{n_i}}\left(x_0\right)-J_{\overline{u}}\left(x_0\right)\right)=0.$$

Now we obtain $\delta\leq 0$ in (9), which is a contradiction. Thus $J_{\overline{u}}\left(\overline{x}\right)=\inf_{y\in E}J_{\overline{u}}\left(y\right)$ and since $J_{\overline{u}}$ is differentiable in the sense of Gâteaux we have $\overline{x}\in V_{\overline{u}}$. Hence $\overline{x}$ necessarily satisfies (2)-(3). On the other hand, if we have $J_{\overline{u}}\left(x_0\right)=J_{\overline{u}}\left(\overline{x}\right)$ the result readily follows. ∎

2.2 Case of positive solutions

It remains to consider the question of the existence and the dependence on parameters for positive solutions. The approach of [10] allows for obtaining at least one positive solution to (2)-(3) with some assumptions added to those leading to the existence result. In fact, the same holds true for the variational formulation although with modified assumptions. We must add some assumption to **A1**, **A2**, **A3** and modify **A4** in assumptions **A1**, **A3, A4**. Namely, we assume that

A5 $f\left(k,y,u\right)-g\left(k\right)\geq 0$ for all $k\in[1,T]$, all $y\in R$ and all $|u|\leq M$; there exists $k_1\in[1,T]$ such that $f\left(k_1,y,u\right)-g\left(k_1\right)>0$ for all $y\in R$ and all $|u|\leq M$;

A6 $\lim_{y\to\infty}\sum_{k=1}^{T}F\left(k,y,u\right)=-\infty$ and $\lim_{y\to-\infty}\sum_{k=1}^{T}F\left(k,y,u\right)=c\in R$ uniformly in $|u|\leq M$.

Remark 2 *Assumption* ***A6*** *is different from* ***A4****. Indeed, function* $F\left(x\right)=-e^{x}$ *satisfies* ***A6*** *and it does not satisfy* ***A4*** *while function* $F\left(x\right)=-x^{l}$ *for any even* l *satisfies* ***A4*** *and it does not satisfy* ***A6****. Still both assumptions* ***A4*** *and* ***A6*** *yield that functional* J_u *is coercive.*

We recall that by a positive solution to (2)-(3) we mean such a function $x \in E$ which satisfies (2) and which is such that $x(k) > 0$ for $k \in [1, T]$ with $x(0) = x(T+1) = 0$. We have the following result concerning positive solutions.

Corollary 3 *Assume either **A1**, **A2**, **A3**, **A5** or **A1**, **A3**, **A5**, **A6**. For any fixed $u \in L_M$ there exists at least one solution $x \in V_u$, $x(k) > 0$ for $k \in [1, T]$, to problem (2)-(3) such that $J_u(x) = \inf_{y \in E} J_u(y)$. Let $\{u_n\}_{n=1}^{\infty} \subset L_M$ be a convergent sequence of parameters, where $\lim_{n\to\infty} u_n = \overline{u} \in L_M$. For any sequence $\{x_n\}_{n=1}^{\infty}$ of positive solutions $x_n \in V_{u_n}$ to the problem (2)-(3) corresponding to u_n, there exist a subsequence $\{x_{n_i}\}_{i=1}^{\infty} \subset E$ and an element $\overline{x} \in E$ such that $\lim_{i\to\infty} x_{n_i} = \overline{x}$ and $J_{\overline{u}}(\overline{x}) = \inf_{y \in E} J_{\overline{u}}(y)$. Moreover, $\overline{x} > 0$ and $\overline{x}$ satisfies (2)-(3) with $\overline{u}$.*

Proof. Since in both cases solutions exist, we need to prove only that the solutions to (2)-(3) under either **A1**, **A2**, **A3**, **A5** or **A1**, **A3**, **A5**, **A6** are positive. We rewrite (2) as follows

$$-\Delta(p(k)\Delta x(k-1)) = f(k, x(k), u(k)) - g(k)$$

and observe that by **A5** we have $-\Delta(p(k)\Delta x(k-1)) \geq 0$. Thus the strong comparison principle, Lemma 2.3 from [1], shows that either $x(k) \geq 0$ for $k \in [1, T]$ or $x(k) = 0$ for $k \in [1, T]$. Since $f(k_1, x(k), u(k)) - g(k_1) \neq 0$ for certain k_1 we cannot have $x = 0$. Thus, we see that $x(k) > 0$ for $k \in [1, T]$. ■

We note that neither in [8] nor in [3] positive solutions are considered. However, in [10] by the lower-upper function method the authors obtain the existence of positive solutions for system (2)-(3) without a parameter with **A1**, **A3** and with assumptions that $g(k) < 0$, $f(t, 0) \geq 0$ for $t \in [1, T]$ (replacing **A5**) and that there exists $\alpha > 0$ such that $f(k, y) \leq 0$ for all $y \geq \alpha$ and $k = 1, ..., T$ (replacing A**2**).

3 Applications for the discrete Emden-Fowler equation

Now we turn to sketching some further possible applications of our results. As an example we shall consider the discrete version of the Emden-Fowler

equation investigated with the aid of critical point theory in [6]. Following the authors of [6] we consider (in R^T with classical Euclidean norm) the discrete equation

$$\Delta\left(p\left(k-1\right)\Delta x\left(k-1\right)\right)+q\left(k\right)x\left(k\right)+f\left(k,x\left(k\right),u\left(k\right)\right)=g\left(k\right) \tag{12}$$

subject to a parameter $u\in L_M$ and with boundary conditions

$$x\left(0\right)=x\left(T\right),\ p\left(0\right)\Delta x\left(0\right)=p\left(T\right)\Delta x\left(T\right). \tag{13}$$

It is assumed that

A7 $f\in C\left(\left[1,T\right]\times R\times\left[-M,M\right],R\right)$, $p\in C\left(\left[1,T+1\right],R\right),q,g\in C\left(\left[1,T\right],R\right)$; $g\left(k_1\right)\neq 0$ for certain $k_1\in\left[1,T\right]$;

A8 there exists a constant $r\in(1,2)$ such that

$$\lim_{|y|\to\infty}\sup\frac{f\left(k,y,u\right)}{\left|y\right|^{r-1}}\leq 0 \tag{14}$$

uniformly for $u\in[-M,M]$, $k\in\left[1,T\right]$.

Basing on ideas developed in the proof of Theorem 1 we formulate and prove the main result of this section. Let us denote

$$M=\begin{bmatrix} p\left(0\right)+p\left(1\right) & -p\left(1\right) & 0 & \dots & 0 & -p\left(0\right)\\ -p\left(1\right) & p\left(1\right)+p\left(2\right) & -p\left(2\right) & \dots & 0 & 0\\ 0 & -p\left(2\right) & p\left(2\right)+p\left(3\right) & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & p\left(T-2\right)+p\left(T-1\right) & -p\left(T-1\right)\\ -p\left(0\right) & 0 & 0 & \dots & -p\left(T-1\right) & p\left(T-1\right)+p\left(0\right) \end{bmatrix}$$

and

$$Q=\begin{bmatrix} -q\left(1\right) & 0 & 0 & \dots & 0 & 0\\ 0 & -q\left(2\right) & 0 & \dots & 0 & 0\\ 0 & 0 & -q\left(3\right) & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -q\left(T-1\right) & 0\\ 0 & 0 & 0 & \dots & 0 & -q\left(T\right) \end{bmatrix}$$

For a fixed $u \in L_M$ we introduce the action functional for (12)-(13)

$$J_u(x) = \frac{1}{2}\langle (M+Q)x, x\rangle - \sum_{k=1}^{T} F(k, x(k), u(k)) + \sum_{k=1}^{T} g(k)x(k).$$

Next, we introduce the set of critical points of (12)-(13)

$$V_u = \left\{ x \in R^T : J_u(x) = \inf_{v \in R^T} J_u(v), \ \frac{d}{dx} J_u(x) = 0 \right\}.$$

Theorem 4 *Assume **A7, A8** and that $M+Q$ is a positive definite matrix. For any fixed $u \in L_M$ there exists at least one non trivial solution $x \in V_u$ to problem (12)-(13). Let $\{u_n\}_{n=1}^{\infty} \subset L_M$ be a convergent sequence of parameters, where $\lim_{n\to\infty} u_n = \overline{u} \in L_M$. For any sequence $\{x_n\}_{n=1}^{\infty}$ of solutions $x_n \in V_{u_n}$ to the problem (12)-(13) corresponding to u_n, there exist a subsequence $\{x_{n_i}\}_{i=1}^{\infty} \subset R^T$ and an element $\overline{x} \in R^T$ such that $\lim_{i\to\infty} x_{n_i} = \overline{x}$ and $\overline{x} \in V_{\overline{u}}$, i.e. $\overline{x}$ satisfies (12)-(13) with $\overline{u}$,*

$$\Delta(p(k-1)\Delta\overline{x}(k-1)) + q(k)\overline{x}(k) + f(k, \overline{x}(k), \overline{u}(k)) = g(k),$$

$$\overline{x}(0) = \overline{x}(T), \ p(0)\Delta\overline{x}(0) = p(T)\Delta\overline{x}(T).$$

Proof. First we must show that for any fixed $u \in L_M$ there exists a solution to (12)-(13) and next we need to show that set V_u is bounded uniformly in $u \in L_M$.

Let us fix $u \in L_M$. By Theorem 3.4 from [6] applied to our functional we get the existence of at least one solution to (12)-(13). Indeed, we recall some arguments used in [6] for convenience. Let us fix $u \in L_M$. Fix $\varepsilon > 0$. By (14), we see that there exists $B > 0$ such that $\frac{f(k,y,u)}{|y|^{r-1}} \leq \varepsilon$ for all $k \in [1,T]$ and for $|y| \geq B$, $|u| \leq M$. Then it follows that $F(k,y,u) \leq \frac{\varepsilon}{r}|y|^r$ for all $k \in [1,T]$ and for $|y| \geq B$, $|u| \leq M$. Denoting $A = \sup_{(k,x,u)\in[1,T]\times[-B,B]\times[-M,M]} |f(k,x,u)|$ we see that for $(k,y,u) \in [1,T] \times R \times [-M,M]$ by the definition of F

$$F(k,y,u) \leq AB + \frac{\varepsilon}{r}|y|^r.$$

Since $M+Q$ is positive definite there exists a number $a_{M+Q} > 0$ such that for all $y \in R^T$

$$\langle (M+Q)y, y\rangle \geq a_{M+Q}|y|^2$$

Therefore, we have by Schwartz inequality for any $y \in R$

$$J_u(y) \geq \frac{1}{2} a_{M+Q} |y|^2 - T\left(AB + \frac{\varepsilon}{r} |y|^r\right) - |y| \sqrt{\sum_{k=1}^{T} g^2(y)}. \tag{15}$$

Since $r < 2$, we see that J_u is coercive. Hence it has an argument of a minimum x which satisfies (12)-(13). We note that $x \neq 0$. Indeed, if $x = 0$, then $g(k_1) = 0$, which is a contradiction with **A12**.

Now we see that by inequality (15) we again have for the solution x_u to (12)-(13)

$$\frac{1}{2} a_{M+Q} |x_u|^2 - T\frac{\varepsilon}{r} |x_u|^r - |x_u| \sqrt{\sum_{k=1}^{T} g^2(y)} \leq J_u(x_u) \leq ABT.$$

Thus the reasoning from the second part of the proof of Theorem 1 now applies. ■

We conclude the paper with some examples and remarks concerning the results obtained in this work.

Example 5 *Let l be any natural number and let $q, r \in C(R, R_+)$ be bounded. Function $f(k,x,u) = q(k)h(x)r(u)$ with*

$$h(x) = \begin{cases} x^{2l}, & x \leq 0 \\ -x^{2l}, & x > 0 \end{cases}$$

does not satisfy ***A2****, but it satisfies* ***A4****.*

Example 6 *Let $q, r \in C(R, R_+)$, where r is a bounded function. Function $f(k,x,u) = q(k)h(x)r(u)$ with*

$$h(x) = \begin{cases} -\frac{x+1}{1+x^4}, & x < 0 \\ -1, & x \geq 0 \end{cases}$$

satisfies ***A6****. Indeed, in this case*

$$H(x) = \begin{cases} \frac{1}{8}\sqrt{2}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + \frac{1}{4}\sqrt{2}\arctan\left(x\sqrt{2}+1\right) + \frac{1}{4}\sqrt{2}\arctan\left(x\sqrt{2}-1\right), & x < 0 \\ -x, & x \geq 0 \end{cases}.$$

Hence ***A6*** *can be directly verified. Taking $g \in C(R, (-\infty, -1))$ we see that* ***A5*** *is also satisfied.*

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